On the Cobordism Class of the Hilbert Scheme of a Surface

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Abstract

Let S be a smooth projective surface and $S^{[n]}$ the Hilbert scheme of zero-dimensional subschemes of S of length n. We proof that the class of $S^{[n]}$ in the complex cobordism ring depends only on the class of the surface itself. Moreover, we compute the cohomology and holomorphic Euler characteristics of certain tautological sheaves on $S^{[n]}$ and prove results on the general structure of certain integrals over polynomials in Chern classes of tautological sheaves.

Let S be a smooth projective surface over the field of complex numbers. For a nonnegative integer n let $S^{[n]}$ denote the Hilbert scheme parameterizing zero-dimensional subschemes of length n. By a well-known result of Fogarty [10] the scheme $S^{[n]}$ is smooth and projective of dimension $2n$, and is irreducible if S is irreducible.

Let $\Omega = \Omega^U \otimes \mathbb{Q}$ be the complex cobordism ring with rational coefficients. Milnor [20] showed that Ω is a polynomial ring freely generated by the cobordism classes $[\mathbb{CP}_i]$ for $i \in \mathbb{N}$. For a smooth and projective complex surface we define

$$H(S) := \sum_{n=0}^{\infty} [S^{[n]}]\, z^n,$$

which is an invertible element in the formal power series ring $\Omega[[z]]$.

The main result of this note is the following

Theorem 0.1 — *$H(S)$ depends only on the cobordism class $[S] \in \Omega_2$.*

From this we have an immediate corollary:

Corollary 0.2 — *If the class of a surface S satisfies the linear relation $[S] = a_1[S_1] + a_2[S_2]$ for two surfaces S_1 and S_2 and two rational numbers a_1 and a_2, then*

$$H(S) = H(S_1)^{a_1} H(S_2)^{a_2}.$$

Proof. Assume first that $a_1 = a_2 = 1$. The class $[S_1] + [S_2]$ is represented by the disjoint union $S_1 \sqcup S_2$. It is clear that the Hilbert scheme of a disjoint union satisfies

$$(S_1 \sqcup S_2)^{[n]} = \coprod_{n_1+n_2=n} S_1^{[n_1]} \times S_2^{[n_2]}. \tag{1}$$

Hence we get $H(S) = H(S_1 \sqcup S_2) = H(S_1)H(S_2)$. By induction, we get $H(S)^n = H(S_1)^{n_1}H(S_2)^{n_2}$ whenever there is a relation $n[S] = n_1[S_1] + n_2[S_2]$ for positive integers n_1, n_2 and n. The corollary follows formally from this. $\square$

If ϕ is any genus, i.e. a ring homomorphism from Ω to another ring [16], this gives $\phi(H(S)) = \phi(H(S_1))^{a_1}\phi(H(S_2))^{a_2}$ whenever $[S] = a_1[S_1] + a_2[S_2]$. Hence if the value of a genus on $H(S)$ is known for two independent surfaces, the value on $H(S)$ is determined for any surface S. Using this we give a new proof of the following theorem first proved in [13].

Theorem 0.3 —

$$\chi_{-y}(H(S)) = \exp\left(\sum_{m=1}^{\infty} \frac{\chi_{-y^m(S)}}{1-(yz)^m}\frac{z^m}{m}\right).$$

Proof. Both sides of this equality are multiplicative in $[S]$, so it suffices to check it for $\mathbb{CP}_2$ and $\mathbb{CP}_1 \times \mathbb{CP}_1$. Now the Hilbert schemes of both $\mathbb{CP}_2$ and $\mathbb{CP}_1 \times \mathbb{CP}_1$ have a $\mathbb{C}^*$-action with isolated fix-points. For such varieties the χ_y-genus is given by

$$\chi_{-y} = \sum_x y^{\dim T(x)^+} = \sum_{p\geq 0} b_{2p}y^p,$$

where the first sum is over all fix-points and $T(x)^+$ is the subspace of the tangent space at x generated by eigenvectors for the $\mathbb{C}^*$-action whose eigenvalues have positive weight. In the second sum the b_j are the Betti numbers of the variety. Recall from [7] that the odd Betti numbers for $\mathbb{CP}_2^{[n]}$ and $(\mathbb{CP}_1 \times \mathbb{CP}_1)^{[n]}$ vanish whereas the even ones are given by

$$b_{2p} = \sum_{n_1+n_2+n_3=n}\ \sum_{r_3-r_1=p-n} p(n_1,r_1)p(n_2)p(n_3,r_3),$$

and

$$b_{2p} = \sum_{n_1+n_2+n_3+n_4=n}\ \sum_{r_4-r_1=p-n} p(n_1,r_1)p(n_2)p(n_3)p(n_4,r_4),$$

where $p(a,b)$ denotes the number of partitions of the nonnegative integer a into b parts. This gives

$$\chi_{-y}(\mathbb{CP}_2^{[n]}) = \sum_p \sum_{n_1+n_2+n_3=n}\ \sum_{r_3-r_1=p-n} p(n_1,r_1)p(n_2)p(n_3,r_3)y^p.$$

and

$$\chi_{-y}((\mathbb{CP}_1 \times \mathbb{CP}_1)^{[n]}) = \sum_p \sum_{n_1+n_2+n_3+n_4=n}\ \sum_{r_4-r_1=p-n} p(n_1,r_1)p(n_2)p(n_3)p(n_4,r_4)y^p.$$

Now easy calculations using that for any integer ε we have

$$\prod_{k\geq 1}\left(1-y^{k+\varepsilon}z^k\right)^{-1}=\sum_{n,r}p(n,r)y^{n+\varepsilon r}z^n$$

show that

$$\chi_{-y}(H(\mathbb{CP}_2))=\prod_{k\geq 1}\left(1-y^{k-1}z^k\right)^{-1}\left(1-y^kz^k\right)^{-1}\left(1-y^{k+1}z^k\right)^{-1}$$

and

$$\chi_{-y}(H(\mathbb{CP}_1\times\mathbb{CP}_1))=\prod_{k\geq 1}\left(1-y^{k-1}z^k\right)^{-1}\left(1-y^kz^k\right)^{-2}\left(1-y^{k+1}z^k\right)^{-1}.$$

On the other hand we have $\chi_{-y}(\mathbb{CP}_2)=1+y+y^2$ and $\chi_{-y}(\mathbb{CP}_1\times\mathbb{CP}_1)=1+2y+y^2$, and a standard calculation shows that these χ_y–genera are related by the exponential expression in the theorem. $\square$

For integers N and k with $0\leq k\leq N$ there is a genus $\phi_{N,k}$ whose characteristic power series is

$$x\frac{e^{-\frac{k}{N}x}}{1-e^{-x}}.$$

If X is a variety whose canonical line bundle has an N-th root L, the genus $\phi_{N,k}$ has a geometric interpretation as $\chi(X,L^{\otimes k})$. For all varieties ϕ is the value of the level-N elliptic genus of Hirzebruch [17] in one of the cusps. We have the following

Theorem 0.4 —

$$\phi_{N,k}(H(S))=\frac{1}{(1-t)^{\phi_{N,k}(S)}}.$$

Proof. For a line bundle L on S let $L_n:=f^*g_*(\otimes_{i=1}^n pr_i^*L)^{\mathfrak{S}_n}$, where $f:S^{[n]}\to S^{(n)}$ and $g:S^n\to S^{(n)}$ are the two natural maps and where pr_i is the projection of S^n onto the i-th factor. We will show later that $\chi(L_n)=\binom{\chi(L)+n-1}{n}$, cf. Lemma 5.1. Assume that S is a surface such that $\omega_S=L^{\otimes N}$ for some line bundle L. Then we claim that the same is true for the Hilbert scheme $S^{[n]}$. Indeed, we have $\omega_{S^{[n]}}=(\omega_S)_n$, and L_n is an N-th root of $\omega_{S^{[n]}}$. This shows the theorem for surfaces having an N-th root of the canonical line bundle. The formula in the theorem being multiplicative, it will be sufficient to find two independent surfaces having this property. Indeed, a K3 surface and a product of two curves of the same genus g such that $N|(2g-2)$ will do. $\square$

The strategy for proving Theorem 0.1 is this: First recall that the cobordism class of a stably complex manifold is completely determined by the collection of its Chern numbers. Thus the theorem is equivalent to the following proposition.

Proposition 0.5 — *For any integer n and any partition λ of $2n$ there is a universal polynomial $P_\lambda \in \mathbb{Q}[z_1, z_2]$ such that the following relation holds for every smooth projective surface S:*

$$c_\lambda(S^{[n]}) = P_\lambda(c_1^2(S), c_2(S)).$$

This proposition will be proved by induction on n.

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1 The geometric set-up

Let $\Sigma_n \subset S^{[n]} \times S$ be the universal family of subschemes parameterized by $S^{[n]}$, and let $I_n \subset \mathcal{O}_{S^{[n]}\times S}$ and $\mathcal{O}_n$ denote its ideal sheaf and structure sheaf, respectively.

$$\begin{array}{ccccc} \Sigma_n & \subset & S^{[n]} \times S & \xrightarrow{q} & S \\ & & {\scriptstyle p}\big\downarrow & & \\ & & S^{[n]}. & & \end{array} \tag{2}$$

The induction step involves the incidence variety $S^{[n,n+1]}$ of all pairs $(\xi, \xi') \in S^{[n]} \times S^{[n+1]}$ satisfying $\xi \subset \xi'$. If ξ' is obtained by extending ξ at the closed point $x \in S$, there is an exact sequence

$$0 \longrightarrow I_{\xi'} \longrightarrow I_\xi \xrightarrow{\lambda} k(x) \longrightarrow 0. \tag{3}$$

Observe that λ defines a point in the fibre of the morphism

$$\sigma = (\phi, \rho) : \mathbb{P}(I_n) \longrightarrow S^{[n]} \times S$$

over (ξ, x). (Here for any coherent sheaf F we denote $\mathbb{P}(F) := Proj(Sym^*(F))$, and $\mathcal{O}_F(1)$ is the tautological quotient line bundle.) Conversely, any point in the fibre of σ defines an extension ξ'. In fact, let $\mathcal{L} := \mathcal{O}_{I_n}(1)$, let $\Gamma \subset \mathbb{P}(I_n) \times S$ be the graph of ρ, and let $j := (\mathrm{id}, \rho)$. Then the kernel of the composite epimorphism

$$\beta : (\phi \times \mathrm{id}_S)^* I_n \longrightarrow (\phi \times \mathrm{id}_S)^* I_n|_\Gamma \cong j_*\sigma^* I_n \longrightarrow j_*\mathcal{L}$$

is an $\mathbb{P}(I_n)$-flat family of ideal sheaves and induces a classifying morphism

$$\psi : \mathbb{P}(I_n) \longrightarrow S^{[n+1]},$$

such that $Ker(\beta) = \psi_S^*(I_{n+1})$. This leads to a scheme theoretic isomorphism $S^{[n,n+1]} \cong \mathbb{P}(I_n)$ and the basic diagram

$$\begin{array}{ccccc} S & \xleftarrow{\rho} & \mathbb{P}(I_n) & \xrightarrow{\psi} & S^{[n+1]} \\ & & \big\downarrow{\scriptstyle \phi} & & \\ & & S^{[n]} & & \end{array} \tag{4}$$

together with two short exact sequences of universal families

$$0 \to \psi_S^* I_{n+1} \to \phi_S^* I_n \to j_* \mathcal{L} \to 0 \tag{5}$$

$$0 \to j_* \mathcal{L} \to \psi_S^* \mathcal{O}_{n+1} \to \phi_S^* \mathcal{O}_n \to 0. \tag{6}$$

Note that $j_* \mathcal{L} = p^* \mathcal{L} \otimes \mathcal{O}_\Sigma = p^* \mathcal{L} \otimes \rho_S^* \mathcal{O}_\Delta$. Here and throughout the paper we will use the short form $f_S := f \times \mathrm{id}_S$ in order to simplify the notations.

Since S is smooth of dimension 2 and I_n is $S^{[n]}$-flat and fibrewise torsion free, there is a resolution of length 1

$$0 \longrightarrow A \longrightarrow B \longrightarrow I_n \longrightarrow 0$$

by locally free $\mathcal{O}_{S^{[n]} \times S}$-sheaves A and B of rank a and $a+1$, respectively. Then $\mathbb{P}(I_n)$ embeds into $\mathbb{P}(B)$ as the zero locus of the homomorphism

$$\pi^* A \longrightarrow \pi^* B \longrightarrow \mathcal{O}_B(1)$$

(where $\pi : \mathbb{P}(B) \to S^{[n]} \times S$ is the projection) such that $\mathcal{L} = \mathcal{O}_B(1)|_{\mathbb{P}(I_n)}$. The incidence variety $S^{[n,n+1]}$ is irreducible [9, 4]. (In fact, it is also smooth, though we will not use this. This result has been independently proven by Cheah [4], Ellingsrud (unpublished) and Tikhomirov [22].) As $S^{[n,n+1]}$ has the correct dimension, it is a locally complete intersection and its fundamental class is given by

$$[\mathbb{P}(I_n)] = c_a(\mathcal{L} \otimes \pi^* A^\vee) \in A_{2n+2}(\mathbb{P}(B)).$$

Lemma 1.1 — *Let $\ell = c_1(\mathcal{L})$. Then for any class $u \in A_*(S^{[n]} \times S)$ one has*

$$\sigma_*(\ell^i) = (-1)^i c_i(\mathcal{O}_n) = (-1)^i c_i(-I_n). \tag{7}$$

Proof. Let $\varepsilon := c_1(\mathcal{O}_B(1))$. Then

$$\begin{aligned}
\sigma_*(\ell^i) &= \pi_*(\varepsilon^i \cdot [\mathbb{P}(I_n)]) = \pi_*(\varepsilon^i c_a(\pi^* A^\vee \otimes \mathcal{O}_B(1)) \\
&= \pi_*(\sum_{k=0}^{a} \varepsilon^{i+a-k} \pi^* c_k(A^\vee)) \\
&= \sum_{k=0}^{a} s_{i-k}(B^\vee) c_k(A^\vee) \\
&= c_i(A^\vee - B^\vee) = (-1)^i c_i(\mathcal{O}_n).
\end{aligned}$$

□

Proposition 1.2 — *Let $f_n : S^{[n]} \longrightarrow S^{(n)}$ denote the Hilbert-Chow morphism from the Hilbert scheme to the symmetric product. Then*

$$(f_n)_* \mathcal{O}_{S^{[n]}} = \mathcal{O}_{S^{(n)}} \quad \textit{and} \quad R^i (f_n)_* \mathcal{O}_{S^{[n]}} = 0 \quad \textit{for all } i > 0.$$

Proof. As f_n is a birational proper morphism of normal varieties, it follows that $(f_n)_* \mathcal{O}_{S^{[n]}} = \mathcal{O}_{S^{(n)}}$. $S^{(n)}$ has rational singularities, as the quotient of a smooth variety by a finite group (see [3], Proposition 4.1). Therefore its resolution $f_n : S^{[n]} \to S^{(n)}$ satifies $R^i (f_n)_* \mathcal{O}_{S^{[n]}} = 0$. □

2 The tangent bundle and tautological bundles

For a smooth projective variety X let $K(X)$ denote the Grothendieck group generated by locally free sheaves, or equivalently, arbitrary coherent sheaves. We will denote a sheaf and its class by the same symbol. $K(X)$ is endowed with a ring structure which for classes of locally free sheaves F and G is given by $F \cdot G := F \otimes G$, and a ring involution $^\vee$, which extends $F \mapsto \mathcal{H}om(F, \mathcal{O}_X)$ for a locally free sheaf. Thus for arbitrary coherent sheaves F, G one has $F^\vee = \sum_i (-1)^i \mathcal{E}xt^i(F, \mathcal{O}_X)$ and $F^\vee \cdot G = \sum_i (-1)^i \mathcal{E}xt^i(F, G)$. For $f: X \to Y$ a morphism of smooth projective varieties there is a push-forward $f_!: K(X) \to K(Y)$; $F \mapsto \sum_i (-1)^i R^i f_*(F)$, and a pullback $f^!$ which for a locally free sheaf G on Y is given by $f^! G = f^* G$. In the paragraphs below, we will use the following well-known fact, which holds in greater generality: let G be an S-flat family of sheaves on $Y \times S$, and let $p: Y \times S \to Y$ denote the projection. If $f: Y' \to Y$ is a morphism of schemes, then $f^!(p_! G) = p_!(f_S^! G)$. Moreover, if the support of G is finite over S, there is an isomorphism of sheaves $f^* p_* G \cong p_* f_S^* G$.

In this section we describe recursion relations for the classes of the tangent sheaf T_n of $S^{[n]}$ and certain tautological sheaves $F^{[n]}$ on $S^{[n]}$ which are defined as follows: For any locally free sheaf F on S let $F^{[n]} := p_*(\mathcal{O}_n \otimes q^* F)$ (with p and q as in diagram (2)). This extends to a group homomorphism $^{[n]}: K(S) \to K(S^{[n]})$. We will also write p and q for the projections of $S^{[n,n+1]} \times S$ to $S^{[n,n+1]}$ and S.

Lemma 2.1 — *The following relation holds in $K(S^{[n,n+1]})$:*

$$\psi^! F^{[n+1]} = \phi^! F^{[n]} + \mathcal{L} \cdot \rho^! F. \tag{8}$$

Proof. Let F be a locally free sheaf on S. Apply the functor $p_*(\,.\, \otimes q^* F)$ to the exact sequence (6) and observe that

$$p_*(\phi_S^*(\mathcal{O}_n) \otimes q^* F) = p_*(\phi_S^*(\mathcal{O}_n \otimes q^* F)) = \phi^*(p_*(\mathcal{O}_n \otimes q^* F)) = \phi^* F^{[n]},$$

i.e. $p_!(\phi_S^!(\mathcal{O}_n) \cdot q^! F) = \phi^! F^{[n]}$ and similarly $p_!(\psi_S^!(\mathcal{O}_{n+1}) \cdot q^! F) = \psi^! F^{[n+1]}$. Using $p \circ j = \mathrm{id}_{S^{[n,n+1]}}$ we get

$$p_*(j_*(\mathcal{L}) \otimes q^* F)) = (p \circ j)_*(\mathcal{L} \otimes (q \circ j)^* F) = \mathcal{L} \otimes \rho^* F,$$

i.e. $p_!(j_!(\mathcal{L}) \cdot q^! F)) = \mathcal{L} \cdot \rho^! F$. □

Next, we turn to the tangent sheaf:

Proposition 2.2 — *The class of the tangent sheaf in $K(S^{[n]})$ is given by the relation*

$$T_n = \chi(\mathcal{O}_S) \cdot 1 - p_!(I_n{}^\vee \cdot I_n). \tag{9}$$

Proof. We have the following isomorphisms

$$T_n \cong \mathrm{Hom}_p(I_n, \mathcal{O}_n) \cong \mathrm{Ext}^1_p(\mathcal{O}_n, \mathcal{O}_n) \cong p_* \mathcal{E}xt^1_{\mathcal{O}_{S^{[n]} \times S}}(\mathcal{O}_n, \mathcal{O}_n).$$

Here Ext_p are the higher derived functors of the composite functor $p_* \circ \mathcal{H}om$. For the first isomorphism see e.g. [19]. The last equality is a consequence from the spectral sequence $R^i p_* \mathcal{E}xt^j \Rightarrow \mathrm{Ext}_p^{i+j}$ and the observation that the sheaves $\mathcal{E}xt^*(\mathcal{O}_n, \mathcal{O}_n)$ are supported on the universal family Σ_n, which is finite over $S^{[n]}$ so that all higher direct images vanish. Moreover,

$$\mathcal{E}xt^0(\mathcal{O}_n, \mathcal{O}_n) \cong \mathcal{E}xt^0(\mathcal{O}_{S^{[n]}\times S}, \mathcal{O}_n) \cong \mathcal{O}_n$$

and, by $\mathcal{E}xt^0(\mathcal{O}_n, \mathcal{O}_{S^{[n]}\times S}) = 0$, $\mathcal{E}xt^1(\mathcal{O}_n, \mathcal{O}_{S^{[n]}\times S}) = 0$ also

$$\mathcal{E}xt^2(\mathcal{O}_n, \mathcal{O}_n) \cong \mathcal{E}xt^2(\mathcal{O}_n, \mathcal{O}_{S^{[n]}\times S}) = \mathcal{O}_n{}^\vee.$$

Hence we get

$$\begin{aligned} T_n &= p_* \mathcal{E}xt^1(\mathcal{O}_n, \mathcal{O}_n) = p_! \mathcal{O}_n - p_!(\mathcal{O}_n{}^\vee \cdot \mathcal{O}_n) + p_! \mathcal{O}_n{}^\vee \\ &= p_!(1) - p_!((1-\mathcal{O}_n)^\vee (1-\mathcal{O}_n)) = \chi(\mathcal{O}_S)\cdot 1 - p_!(I_n{}^\vee \cdot I_n) \end{aligned}$$

as an identity in $K(S^{[n]})$. □

Proposition 2.3 — *The following relation holds in* $K(S^{[n,n+1]})$*:*

$$\psi^! T_{n+1} = \phi^! T_n + \mathcal{L}\cdot \sigma^! I_n{}^\vee + \mathcal{L}^\vee \cdot \sigma^! I_n \cdot \rho^! \omega_S{}^\vee - \rho^!(1 - T_S + \omega_S{}^\vee). \tag{10}$$

Proof. We have

$$\begin{aligned} \psi^! T_{n+1} &= \psi^!(\chi(\mathcal{O}_S)\cdot 1) - \psi^! p_!(I_{n+1}{}^\vee \cdot I_{n+1}) \\ &= \phi^!(\chi(\mathcal{O}_S)\cdot 1) - p_!(\psi_S^! I_{n+1}{}^\vee \cdot \psi_S^! I_{n+1}) \end{aligned}$$

Now use (5) to replace $\psi_S^! I_{n+1}$ by $\phi_S^! I_n - p^! \mathcal{L}\cdot \rho_S^! \mathcal{O}_\Delta$, where $\Delta \subset S\times S$ denotes the diagonal. This yields

$$\begin{aligned} \psi^! T_{n+1} &= \phi^! T_n - \rho^! p_!(\mathcal{O}_\Delta{}^\vee \cdot \mathcal{O}_\Delta) \\ &\quad + p_!(\rho_S^! \mathcal{O}_\Delta \cdot \phi_S^! I_n{}^\vee)\cdot \mathcal{L} + p_!(\rho_S^! \mathcal{O}_\Delta{}^\vee \cdot \phi_S^! I_n). \end{aligned}$$

The two last summands can be simplified as follows:

$$p_!(\rho_S^! \mathcal{O}_\Delta \cdot \phi_S^! I_n{}^\vee) = p_!(j_!(j^! \phi_S^! I_n{}^\vee)) = p_! j_! \sigma^! I_n{}^\vee = \sigma^! I_n{}^\vee$$

and similarly $p_!(\rho_S^! \mathcal{O}_\Delta{}^\vee \cdot \phi_S^! I_n) = \sigma^! I_n \cdot \rho^! \omega_S{}^\vee$, since $\rho_S^! \mathcal{O}_\Delta{}^\vee = \rho_S^! \Delta_! \omega_S{}^\vee = j_! \rho^! \omega^\vee$. Finally, $p_!(\mathcal{O}_\Delta{}^\vee \cdot \mathcal{O}_\Delta) = \mathcal{O}_S - T_S + \omega_S{}^\vee$, which follows e.g. by Proposition 2.2. □

3 The induction step

We want to relate integrals on $S^{[n+1]}\times S^m$ to integrals on $S^{[n]}\times S^{m+1}$.

Let $Z = S^{[n,n+1]}\times S^m$. The maps ϕ and ψ from diagram (4) generalize to morphisms $\Psi = \psi \times \mathrm{id}_{S^m} : Z \to S^{[n+1]}\times S^m$ and $\Phi = \sigma\times \mathrm{id}_{S^m} : Z \to S^{[n]}\times S^{m+1}$. For any $I \subset \{0,1,\ldots,m\}$ let pr_I denote the projection from $S^{[n+1]}\times S\times \ldots \times S$ to the product of the factors indexed by I.

Proposition 3.1 — *Let f be a polynomial in the Chern classes of the following sheaves on $S^{[n+1]} \times S^m$:*

$$pr_0^* T_{n+1}, \;\; pr_{0i}^* I_{n+1}, \;\; pr_{ij}^* \mathcal{O}_\Delta, \;\; pr_i^* T_S \;\; \text{for any } 1 \leq i, j \leq m.$$

Then there is a polynomial $\tilde{f}$, depending only on f, in the Chern classes of the analogously defined sheaves on $S^{[n]} \times S^{m+1}$ such that

$$\int_{S^{[n+1]} \times S^m} f = \int_{S^{[n]} \times S^{m+1}} \tilde{f}.$$

Proof. The morphism Ψ is generically finite of degree $n+1$, so that

$$\int_{S^{[n+1]} \times S^m} f = \frac{1}{n+1} \int_Z \Psi^* f.$$

Because of an index shift resulting from the insertion of the additional factor S between $S^{[n]}$ and S^m we have

$$\Psi^! pr_i^* T_S = \Phi^! pr_{i+1}^* T_S, \; \Psi^! pr_{ij}^* \mathcal{O}_\Delta = \Phi^! pr_{i+1,j+1}^* \mathcal{O}_\Delta.$$

Using (5) we get

$$\Psi^! pr_{0i}^* I_{n+1} = \Phi^! pr_{0,i+1}^* I_n + pr_Z^! \mathcal{L} \cdot pr_{1,i+1}^* \mathcal{O}_\Delta.$$

And finally by (10)

$$\begin{aligned} \Psi^! pr_0^* T_{n+1} \quad = \quad & \Phi^! pr_0^* T_n + pr_{S^{[n,n+1]}}^! \mathcal{L} \cdot pr_{01}^* I_n{}^\vee & (11) \\ & + pr_{S^{[n,n+1]}}^! \mathcal{L}^\vee \cdot pr_{01}^! I_n \cdot pr_1^! \omega_S{}^\vee - pr_1^! (\mathcal{O}_S - T_S + \omega_S{}^\vee). & (12) \end{aligned}$$

It follows that there are polynomials f_ν depending only on f, in the Chern classes of the sheaves

$$pr_0^* T_n, \, pr_{0i}^* I_n, \, pr_{ij}^* \mathcal{O}_\Delta, \, pr_i^* T_S$$

such that

$$\int_{S^{[n+1]} \times S^{[m]}} f = \frac{1}{n+1} \int_Z \Psi^* f = \int_Z \big(\sum_{\nu \geq 0} \Phi^* f_\nu \cdot pr_Z^* (-c_1(\mathcal{L}))^\nu \big).$$

As we are only trying to prove a general structure result we make no attempt to derive from the above recursion relations for the classes in the K-groups more explicit formulae for the dependence of f_ν on f.

Now, according to (7), the last integral equals:

$$\int_{S^{[n]} \times S^{m+1}} \Phi_* \big(\sum_{\nu \geq 0} \Phi^* f_\nu \cdot pr_Z^* (-c_1(\mathcal{L}))^\nu \big) = \int_{S^{[n]} \times S^{m+1}} \sum_{\nu \geq 0} f_\nu \cdot c_\nu (-pr_{01}^* I_n).$$

The integrand in this expression is the polynomial $\tilde{f}$. $\square$

Proof of Proposition 0.5. Suppose we are given a polynomial P in the Chern classes of T_n. Applying the proposition repeatedly, we may write

$$\int_{S^{[n]}} P = \int_{S^n} \tilde{P}$$

for some polynomial $\tilde{P}$, which depends only on P, in the Chern classes of sheaves on S^n of the form $pr_i^* T_S$ and $pr_{ij}^* \mathcal{O}_\Delta$. Any such expression $\int_{S^n} \tilde{P}$ can be universally reduced to a polynomial expression of integrals of polynomials in the Chern classes of T_S (to see this for the Chern classes of $pr_{ij}^* \mathcal{O}_\Delta$ one applies Riemann-Roch without denominators [14]). This finishes the proof. $\square$

4 A generalization

We can generalize the methods used above to prove Proposition 0.5 to cover integrals of polynomial expressions in the Chern classes of tautological sheaves. Let $F_1, \ldots, F_\ell \in K(S)$. We require that the rank r_i of F_i $(i = 1, \ldots, \ell)$ is the same on all connected components of S.

Theorem 4.1 — *Let P be a polynomial in the Chern classes of the tangent bundle T_n of $S^{[n]}$ and the Chern classes of $F_1^{[n]}, \ldots, F_\ell^{[n]}$. Then there is a universal polynomial $\widetilde{P}$, depending only on P, in the Chern classes of T_S, the $r_1, \ldots, r_k$ and the Chern classes of $F_1, \ldots, F_\ell$ such that*

$$\int_{S^{[n]}} P = \int_S \widetilde{P}.$$

Proof. The proof goes along similar lines as that of Proposition 0.5. The result immediately follows from a modified version of Proposition 3.1: We now allow f to be a polynomial in the r_k and the Chern classes of the

$$pr_0^* F_k^{[n+1]}, pr_0^* T_{n+1}, pr_{0i}^* I_{n+1}, pr_{ij}^* \mathcal{O}_\Delta, pr_i^* F_k, pr_i^* T_S$$

on $S^{[n+1]} \times S^m$, and get $\widetilde{f}$ to be a polynomial in the r_k and the Chern classes of the analogously defined bundles on $S^{[n]} \times S^{m+1}$. To prove this use the recursion relation (8) and the formula

$$c_i(\mathcal{L} \otimes \rho^* F_k) = \sum_{a+b=i} \binom{r_k - b}{a} c_1(\mathcal{L})^a \rho^*(c_b(F_k)),$$

to obtain

$$\int_{S^{[n+1]} \times S^m} f = \int_Z \big(\sum_{\nu \geq 0} \Phi^* f_\nu \cdot pr_Z^*(-c_1(\mathcal{L}))^\nu \big),$$

where the f_ν are universal polynomials in the r_k and the Chern classes of

$$pr_0^* F_k^{[n]}, pr_0^* T_n, pr_{0i}^* I_n, pr_{ij}^* \mathcal{O}_\Delta, pr_i^* F_k, pr_i^* T_S.$$

Then we again use (7). $\square$

We can use Theorem 4.1 to make predictions on the algebraic structure of certain formulae: let $\Psi : K \longrightarrow H^\times$ be a multiplicative function, i.e. for any complex manifold X we are given a group homomorphism from the additive group $K(X)$ into the multiplicative group of units in $H(X;\mathbb{Q})$, which is functorial with respect to pull-backs and is polynomial in the Chern classes of its argument. The total Chern class, the total Segre class or the Chern character of the determinant are of this type. Also let $\phi(x) \in \mathbb{Q}[[x]]$ be a formal power series and, for a complex manifold X of dimension n put $\Phi(X) := \phi(x_1)\cdot\ldots\cdot\phi(x_n) \in H^*(X,\mathbb{Q})$ with $x_1,\ldots,x_n$ the Chern roots of T_X.

Let S be a smooth projective surface and let $x \in K(S)$. For any functions Φ and Ψ as above we define a power series in $\mathbb{Q}[[z]]$ as follows:

$$H_{\Psi,\Phi}(S,x) := \sum_{n=0}^{\infty} \int_{S^{[n]}} \Psi(x^{[n]})\Phi(S^{[n]})\, z^n.$$

Theorem 4.2 — *For each integer r there are universal power series $A_i \in \mathbb{Q}[[z]]$, $i = 1,\ldots,5$, depending only on Ψ, Φ and r, such that for each $x \in K(S)$ of rank r (on every component of S) one has*

$$H_{\Psi,\Phi}(S,x) = \exp(c_1^2(x)A_1 + c_2(x)A_2 + c_1(x)c_1(S)A_3 + c_1^2(S)A_4 + c_2(S)A_5).$$

For simplicity we have suppressed the integrals $\int_S$ in the statement of the theorem and interpret the expressions $c_1(x)c_1(S)$ etc. as intersection numbers.

Proof. Let $\mathcal{K}_r := \{(S,x)|S \text{ an algebraic surface}, x \in K(S), \mathrm{rank}(x) = r\}$, and let $\gamma : \mathcal{K}_r \to \mathbb{Q}^5$ be the map $(S,x) \mapsto (c_1^2(x), c_2(x), c_1(x)c_1(S), c_1^2(S), c_2(S))$. The images of the five elements $(\mathbb{CP}_2, r\cdot 1)$, $(\mathbb{CP}_2, \mathcal{O}(1)+(r-1)\cdot 1$, $(\mathbb{CP}_2, \mathcal{O}(2)+(r-1)\cdot 1)$, $(\mathbb{CP}_2, 2\mathcal{O}(1)+(r-2)\cdot 1)$, and $(\mathbb{CP}_1^2, r\cdot 1)$ under γ are the linearly independent vectors $(0,0,0,9,3)$, $(1,0,3,9,3)$, $(4,0,6,9,3)$, $(4,1,6,9,3)$, and $(0,0,0,8,4)$.

Now, if $S = S_1 \sqcup S_2$ we may decompose $(S,x) \in \mathcal{K}_r$ as $(S_1,x_1)+(S_2,x_2)$, where $x_i = x|_{S_i}$, and get $\gamma(S,x) = \gamma(S_1,x_1)+\gamma(S_2,x_2)$. Moreover, there is a decomposition of the class of the tautological sheaf analogous to the decomposition (1) of the Hilbert scheme:

$$x^{[n]}|_{S_1^{[n_1]}\times S_2^{[n_2]}} = pr_1^*(x_1^{[n_1]}) \cdot pr_2^*(x_2^{[n_2]}).$$

From the multiplicative behaviour of Ψ and Φ we deduce

$$\int_{S^{[n]}} \Psi(x^{[n]})\Phi(S^{[n]}) = \sum_{n_1+n_2=n} \int_{S_1^{[n_1]}} \Psi(x_1^{[n_1]})\Phi(S_1^{[n_1]}) \int_{S_2^{[n_2]}} \Psi(x_2^{[n_2]})\Phi(S_2^{[n_2]})$$

and get

$$H_{\Psi,\Phi}(S,x) = H_{\Psi,\Phi}(S_1,x_1)H_{\Psi,\Phi}(S_2,x_2). \tag{13}$$

By Theorem 4.1, the function $H_{\Psi,\Phi} : \mathcal{K}_r \to \mathbb{Q}[[z]]$ factors through γ and a map $h : \mathbb{Q}^5 \longrightarrow \mathbb{Q}[[z]]$. As the image of γ is Zariski dense in $\mathbb{Q}^5$ we conclude from (13), that

$$\log h(y_1+y_2) = \log h(y_1) + \log h(y_2), \text{ for all } y_1, y_2 \in \mathbb{Q}^5,$$

i.e. $\log h$ is a linear function. This proves the theorem. $\square$

5 Riemann-Roch numbers

In this section we want to compute the Riemann-Roch numbers $\chi(M)$ for line bundles and vector bundles on $S^{[n]}$. Let $L_n := f^*g_*(\otimes_{i=1}^n pr_i^* L)^{\mathfrak{S}_n}$, for L a line bundle on S, where $f : S^{[n]} \to S^{(n)}$ and $g : S^n \to S^{(n)}$ are the natural morphisms, and $pr_i : S^n \to S$ is the i-th projection. This gives a monomorphism $-_n : Pic(S) \to \mathrm{Pic}(S^{[n]}), L \mapsto L_n$. It is well known that for $n \geq 2$

$$Pic(S^{[n]}) = (Pic(S))_n \oplus \mathbb{Z}E, \qquad E := \det(\mathcal{O}_S^{[n]}),$$

($E = \mathcal{O}_S$ in case $n \leq 1$). Moreover, $c_1(E) = -\frac{1}{2}D$, where D is the exceptional divisor of $S^{[n]} \to S^{(n)}$. Note that for $F \in K(S)$

$$\det(F^{[n]}) = \det(F)_n \otimes E^{\mathrm{rk}(F)}.$$

Lemma 5.1 — *The Euler characteristics of $L_n \otimes E$, L_n and L are related by the formulae*

$$\chi(L_n) = \binom{\chi(L) + n - 1}{n} \quad \text{and} \quad \chi(L_n \otimes E) = \binom{\chi(L)}{n}.$$

Proof. Consider the cartesian diagram

$$\begin{array}{ccc} \widehat{S}^n_* & \xrightarrow{\widehat{f}} & S^n_* \\ \widehat{g}\downarrow & & \downarrow g \\ S^{[n]}_* & \xrightarrow{f} & S^{(n)}_*, \end{array} \tag{14}$$

where $S^{(n)}_*$ is the open subscheme of zero cycles of length n whose support consists of at least $(n-1)$ points and $S^{[n]}_*$ and S^n_* are the preimages of $S^{(n)}_*$ under f and g. It is easy to see that $\widehat{S}^n_*$ is the blow-up of S^n_* along the (disjoint) diagonals

$$\Delta_{i,j} = \{(x_1, \ldots, x_n) \in S^n_* \mid x_i = x_j\}.$$

Let $\widehat{\Delta}_{i,j}$ denote the corresponding exceptional divisors, and $\widehat{\Delta} := \sum_{i<j} \widehat{\Delta}_{i,j}$ their sum. The family $\Gamma \subset \widehat{S}^n_* \times S$ corresponding to the classifying morphism $\widehat{g}$ is the union $\bigcup_{i=1}^n \Gamma_i$ of the graphs Γ_i of the i-th projection $\widehat{S}^n_* \to S^n_* \to S$. Projecting the short exact sequence

$$0 \longrightarrow \mathcal{O}_\Gamma \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\Gamma_i} \longrightarrow \bigoplus_{i<j} \mathcal{O}_{\Gamma_i \cap \Gamma_j} \longrightarrow 0$$

to the base $\widehat{S}^n_*$ of these families we obtain a short exact sequence

$$0 \longrightarrow p_*\mathcal{O}_\Gamma \longrightarrow \mathcal{O}_{\widehat{S}^n_*}^{\oplus n} \longrightarrow \bigoplus_{i<j} \mathcal{O}_{\hat{\Delta}_{ij}} = \mathcal{O}_{\hat{\Delta}} \longrightarrow 0$$

of $\mathfrak{S}_n$-linearized sheaves. $\mathfrak{S}_n$ acts on the sheaf in the middle by permutation of the factors. If we take the determinant of the first homomorphism in this sequence, we obtain a short exact sequence

$$0 \longrightarrow \det(p_*\mathcal{O}_\Gamma) \longrightarrow \mathcal{O}^{\varepsilon}_{\widehat{S}^n_*} \longrightarrow \mathcal{O}_{\widehat{\Delta}} \longrightarrow 0$$

of $\mathfrak{S}_n$-linearized sheaves with equivariant homomorphisms, where the upper index ε indicates that the $\mathfrak{S}_n$-linearization of $\mathcal{O}_{\widehat{S}^n_*}$ is the standard one twisted by the alternating character $\varepsilon : \mathfrak{S}_n \to \mathbb{Z}/2$. (Recall that any two G-linearizations of a line bundle on a G-scheme differ by a character of the group G). Thus we can identify $\widehat{g}^*E = \det(p_*\mathcal{O}_\Gamma)$ with the $\mathfrak{S}_n$-linearized subsheaf $\mathcal{O}_{\widehat{S}^n_*}(-\widehat{\Delta})^{\varepsilon} \subset \mathcal{O}^{\varepsilon}_{\widehat{S}^n_*}$, endowed with the alternating linearization.

Now $\widehat{g}^*L_n = \widehat{f}^*L^{\boxtimes n}$ and $\widehat{g}^*(L_n \otimes E) = \widehat{f}^*L^{\boxtimes n} \otimes \mathcal{O}(-\widehat{\Delta}))$. As $S^{[n]}$ is smooth and $S^{[n]} \setminus S^{[n]}_*$ has codimension 2, this gives

$$\begin{aligned} H^0(S^{[n]}, L_n) &= H^0(S^{[n]}_*, L_n) = H^0(\widehat{S}^n_*, \widehat{f}^*L^{\boxtimes n})^{\mathfrak{S}_n} \\ &= H^0(S^n_*, L^{\boxtimes n})^{\mathfrak{S}_n} = H^0(S^n, L^{\boxtimes n})^{\mathfrak{S}_n} \\ &= (H^0(S, L)^{\otimes n})^{\mathfrak{S}_n} \cong S^n H^0(S, L), \end{aligned}$$

and, similarly,

$$\begin{aligned} H^0(S^{[n]}, L_n \otimes E) &\cong H^0(S^{[n]}_*, L_n \otimes E)) \\ &= H^0(\widehat{S}^n_*, \widehat{f}^*L^{\boxtimes n} \otimes \mathcal{O}(-\widehat{\Delta}))^{\mathfrak{S}_n} \\ &= H^0(S^n_*, L^{\boxtimes})^{\mathfrak{S}_n, \varepsilon} = (H^0(S, L)^{\otimes n})^{\mathfrak{S}_n, \varepsilon} \\ &\cong \Lambda^n(H^0(S, L)). \end{aligned}$$

Here the notation $V^{\mathfrak{S}_n, \varepsilon}$ means $\{v \in V | \pi(v) = \varepsilon(\pi) \cdot v, \forall \pi \in \mathfrak{S}_n\}$. Taking dimensions, we get

$$h^0(L_n) = \binom{h^0(L) + n - 1}{n} \quad \text{and} \quad h^0(L_n \otimes E) = \binom{h^0(L)}{n}. \tag{15}$$

Now let H be an ample line bundle on S. By Grothendieck's construction of the Hilbert scheme it follows that $H_n \otimes E \cong \det(H^{[n]})$ is very ample for sufficiently ample H on S. Applying this to $L \otimes H^k$ for sufficiently large k, We conclude that

$$\chi(L_n \otimes H_n^k \otimes E) = h^0(L_n \otimes H_n^k \otimes E) = \binom{h^0(L \otimes H^k)}{n} = \binom{\chi(L \otimes H^k)}{n}.$$

Evaluating this equation of polynomials in k at $k = 0$ one finds $\chi(L \otimes E) = \binom{\chi(L)}{n}$ for all line bundles L.

Let L be an arbitrary line bundle on S and let H be ample S. Then $H^{\boxtimes n}$ is an $\mathfrak{S}_n$-linearized ample line bundle on S^n and descends to an ample line bundle $H^{(n)}$ on $S^{(n)}$. For sufficiently large k, the line bundles $L \otimes H^k$ on S and $(L \otimes H^k \otimes \omega_S{}^\vee)^{(n)}$ on $S^{[n]}$ will be very ample. In particular, $f^*(L \otimes H^k \otimes \omega_S{}^\vee)^{(n)} \cong$

$L_n \otimes H_n^k \otimes \omega_{S^{[n]}}{}^\vee$ is globally generated and big. It follows from the Grauert-Riemenschneider vanishing theorem [15] that $H^i(S^{[n]}, L_n \otimes H_n^k) = 0$ for all $i > 0$. This gives

$$\chi(L_n \otimes H_n^k) = h^0(L_n \otimes H_n^k) = \binom{h^0(L \otimes H^k) + n - 1}{n} = \binom{\chi(L \otimes H^k) + n - 1}{n}$$

for all sufficiently large k. As both sides are polynomials in k, we may take $k = 0$. □

Consider the following power series in $\mathbb{Q}[a, y][[z]]$:

$$f_{y,a} := \sum_{n \geq 0} \binom{y - a(n-1)}{n} z^n, \quad g_{y,a} := \sum_{n \geq 0} \frac{y}{y - an} \binom{y - an}{n} z^n.$$

$f_{y,a}$ and $g_{y,a}$ play a role in combinatorics: Take k points on a line with a distance of 1 from one to the next. Then $\binom{k-a(n-1)}{n}$ is the number of ways to choose n of them with a distance $> a$ among each other, and $\frac{k}{k-an}\binom{k-an}{n}$ is the same number if the k points lie on a circle [21].

In the proof of the following lemma we make use of the Bürmann-Lagrange expansion formula in the following form: if $L(z)$ is a power series, and $\alpha(z)$ is a power series with $\alpha(0) = \alpha'(0) = 0$, then

$$\sum_{n \geq 0} \frac{1}{n!} \frac{d^n}{dz^n} (L(z)\alpha(z)^n) = \left. \frac{L(\zeta)}{1 - \alpha'(\zeta)} \right|_{\zeta = z - \alpha(\zeta)}.$$

For a proof in an analytic context see [23, p 128-135].

Lemma 5.2 — *The power series $f_{y,a}$ and $g_{y,a}$ are related as follows:*

$$g'_{y,a} = y f_{y-2a-1,a}, \qquad g_{y,a} = g_{1,a}^y, \qquad f_{y,a} = g_{1,a}^y \cdot f_{0,a}.$$

Proof. Using new formal variables u, x and v that are related by $z = u^a$, $u = x + x^{a+1}$, and $v = x^a$, so that $z = v(1+v)^a$, and, in the last line of the computation, the Bürmann-Lagrange formula, we have

$$\begin{aligned} f_{y,a}(z) &= \sum_{n \geq 0} \binom{y - a(n-1)}{n} z^n \\ &= \sum_{n \geq 0} \binom{(a+1)n - (y+a+1)}{n} (-z)^n \\ &= \sum_{n \geq 0} u^{y+a+1} \frac{1}{n!} \frac{d^n}{du^n} \frac{(-u^{a+1})^n}{u^{y+a+1}} \\ &= \left(\frac{u}{x}\right)^{y+a+1} \frac{1}{1 + (a+1)x^a} = (1+v)^y \cdot \frac{(1+v)^{a+1}}{1 + (a+1)v} \end{aligned}$$

This gives $f_{0,a} = (1+v)^{a+1}/(1+(a+1)v)$ and $f_{y,a}(z) = f_{0,a}(z) \cdot (1 + v(z))^y$. Now $dz = (1 + (a+1)v)(1+v)^{a-1} dv$ and hence

$$\frac{d}{dz}(1 + v(z)) = \frac{1}{(1+v)^{a-1}(1 + (a+1)v)} = \frac{f_{0,a}}{(1+v)^{2a}}.$$

Differentiating $g_{y,a}$ we find

$$\begin{aligned}
\frac{d}{dz} g_{y,a}(z) &= \sum_{n\geq 1} \frac{yn}{y-an}\binom{y-an}{n} z^{n-1} \\
&= \sum_{n\geq 1} y\binom{y-an-1}{n-1} z^{n-1} \\
&= y f_{y-2a-1,a} = y(1+v)^{y-1-2a} f_{0,a} \\
&= y(1+v)^{y-1} \cdot \frac{f_{0,a}}{(1+v)^{2a}} = \frac{d}{dz}(1+v)^y.
\end{aligned}$$

As both $g_{y,a}$ and $(1+v)^y$ are power series in z with constant term 1, we have $g_{y,a} = (1+v)^y$. Collecting the proven relations we conclude that $g_{1,a} = 1+v$, $g_{y,a} = g_{1,a}^y$ and $f_{y,a} = g_{1,a}^y f_{0,a}$. □

Theorem 5.3 — *Let K denote the canonical divisor of S. For each $r \in \mathbb{Z}$, there exist universal power series $A_r, B_r \in \mathbb{Q}[[z]]$, such that for all $L \in Pic(S)$*

$$\sum_{n\geq 0} \chi(L_n \otimes E^r))z^n = g_{1,r^2-1}(z)^{\chi(L)} \cdot f_{0,r^2-1}(z)^{\frac{\chi(\mathcal{O}_S)}{2}} \cdot A_r(z)^{KL-\frac{K^2}{2}} \cdot B_r(z)^{K^2}.$$

Moreover, A and B satisfy the symmetry relations $A_{-r} = 1/A_r$ and $B_{-r} = B_r$ for arbitrary r, and $A_r = B_r = 1$ for $r = -1, 0, 1$. In particular,

$$\chi(L_n \otimes E^{\pm 1}) = \binom{\chi(L)}{n}.$$

If S is a K3-surface, then

$$\chi(L_n \otimes E^r) = \binom{\chi(L) - (r^2-1)(n-1)}{n},$$

and if S is an abelian surface, then

$$\chi(L_n \otimes E^r) = \frac{\chi(L)}{n}\binom{\chi(L) - (r^2-1)n - 1}{n-1}.$$

Proof. Let $F = L + r \cdot 1 \in K(S)$. Then $\det(F^{[n]}) = L_n \otimes E^r$, $c_1(F) = c_1(L)$, $c_2(F) = 0$. Therefore, by Theorem 4.2,

$$\begin{aligned}
\sum_{n\geq 0} \chi(L_n \otimes E^r) z^n &= Z_1(z)^{K^2} Z_2(z)^{KL} Z_3(z)^{c_2(S)} Z_4(z)^{L^2} \\
&= A_r(z)^{KL-\frac{K^2}{2}} B_r(z)^{K^2} F_r(z)^{\frac{\chi(\mathcal{O}_S)}{2}} G_r(z)^{\chi(L)},
\end{aligned}$$

for suitable power series $Z_i, A_r, B_r, G_r, F_r \in \mathbb{Q}[[z]]$. For the second equality we have used the identities $\chi(L) = \frac{L(L-K)}{2} + \chi(\mathcal{O}_S)$ and $\chi(\mathcal{O}_S) = \frac{c_1(S)^2 + c_2(S)}{12}$.

It is well-known that $\omega_{S^{[n]}} = (\omega_S)_n$. We get by Serre duality

$$\chi(L_n \otimes E^{-r})) = \chi(\omega_{S^{[n]}} \otimes {L_n}^\vee \otimes E^r) = \chi((\omega_S \otimes L^\vee)_n \otimes E^r)).$$

Using $\chi(L) = \chi(\omega_S \otimes L^\vee)$ and $K(K-L) - \frac{K^2}{2} = -(KL - \frac{K^2}{2})$, this gives $B_{-r} = B_r$, $F_{-r} = F_r$, $G_{-r} = G_r$ and $A_{-r} = 1/A_r$.

To determine F_r and G_r explicitly, let S be a K3-surface. Then by [1] the Hilbert scheme $X := S^{[n]}$ is an irreducible symplectic complex manifold.

There exists a natural quadratic form q on $H^2(X,\mathbb{Z})$ (see [1],[11],[18]), which on $H^{1,1}(X)$ is defined as follows: let $\omega \in H^2(X,\mathbb{Z})$ be the everywhere non-degenerate holomorphic 2-form, normalized by $\int_X \omega^n \bar{\omega}^n = n!$. Then for $\alpha \in H^{1,1}(X)$:

$$q(\alpha) = \frac{1}{(n-1)!} \int_X (\omega\bar{\omega})^{n-1} \alpha^2.$$

Moreover, there exists a universal polynomial $h \in \mathbb{Q}[z]$ such that $\chi(M) = h(q(c_1(M)))$ for all M in $Pic(X)$. For $L \in Pic(S)$ Beauville [1, Lemma 9.1, 9.2] showed that $q(c_1(L_n)) = L^2$, $q(c_1(E)) = -2(n-1)$, and E and L_n are orthogonal with respect to q; thus $q(c_1(L_n \otimes E^r)) = L^2 - 2r^2(n-1)$.

The polynomial h is determined by the formula of Lemma 5.1

$$\chi(L_n) = \binom{\chi(L) + n - 1}{n} = \binom{L^2/2 + n + 1}{n} = \binom{q(L)/2 + n + 1}{n}$$

applied to sufficiently many L on S with distinct values of L^2:

$$\begin{aligned} \chi(L_n \otimes E^r) &= h(q(c_1(L_n \otimes E^r))) \\ &= \binom{q(c_1(L_n \otimes E^r))/2 + n + 1}{n} \\ &= \binom{\chi(L) - (r^2-1)(n-1)}{n}. \end{aligned}$$

As $\chi(\mathcal{O}_S) = 2$, we get $F_r \cdot G_r^{\chi(L)} = f_{\chi(L), r^2-1}$. It follows from Lemma 5.2 that we can identify $F_r = f_{0,r^2-1}$ and $G_r = g_{1,r^2-1}$. Finally, the computations of Lemma 5.1 show that the equation $\chi(L \otimes E^r) = \binom{\chi(L)-(r^2-1)(n-1)}{n}$ holds for all surfaces, if we restrict to small ranks $r = -1, 0, 1$. This implies that $A_r = B_r = 1$ for $r = -1, 0, 1$. $\square$

Remark 5.4 — Using Bott's residue formula the coefficients of A_r and B_r can be determined. We computed A_r and B_r up to order 8 in z. Up to order 5 in z we get

$$\begin{aligned} \log(A_r) \equiv\ & \left(\frac{1}{6}r - \frac{1}{6}r^3\right) z^2 + \left(\frac{1}{5}r - \frac{5}{8}r^3 + \frac{17}{40}r^5\right) z^3 \\ & + \left(\frac{29}{140}r - \frac{209}{180}r^3 + \frac{88}{45}r^5 - \frac{631}{630}r^7\right) z^4 \\ & + \left(\frac{13}{63}r - \frac{31259}{18144}r^3 + \frac{16979}{3456}r^5 - \frac{69619}{12096}r^7 + \frac{171215}{72576}r^9\right) z^5 \end{aligned}$$

$$
\begin{aligned}
B_r \quad \equiv \quad & 1 + \left(\frac{1}{24}r^2 - \frac{1}{24}r^4\right) z^2 + \left(\frac{29}{360}r^2 - \frac{31}{144}r^4 + \frac{97}{720}r^6\right) z^3 \\
& + \left(\frac{139}{1260}r^2 - \frac{3053}{5760}r^4 + \frac{2273}{2880}r^6 - \frac{14899}{40320}r^8\right) z^4 \\
& + \left(\frac{187}{1400}r^2 - \frac{6257}{6480}r^4 + \frac{421267}{172800}r^6 - \frac{311701}{120960}r^8 + \frac{503377}{518400}r^{10}\right) z^5
\end{aligned}
$$

The method for showing this result is as follows. By Theorem 5.3 it is enough to compute $\chi((kH)_n + rE)$ for $S = \mathbb{CP}_2$ and H the hyperplane bundle. By the Riemann-Roch theorem we have to compute

$$\int_{\mathbb{CP}_2^{[n]}} td_{\mathbb{CP}_2^{[n]}} \cdot \exp((kH)_n + rE).$$

For this we use Bott's residue formula. The maximal torus Γ of $SL(2,\mathbb{C})$ acts on $\mathbb{CP}_2^{[n]}$ with finitely many fix points Z, which together with the structure of the tangent space as Γ-module are explicitly described in [7]. In [8] also the structure of the fibres $(nH)^{[n]}$ as Γ module is given. This information is enough to apply Bott's residue formula to compute the above intersection number following the strategy outlined in [8]. The computations are carried out with a suitable Maple program.

Remark 5.5 Theorem 0.1 allows us to use the Bott residue formula to compute the Chern numbers of $S^{[n]}$ for all n and any surface S: We computed them for $n \leq 7$. As an illustration we give the numbers for S a $K3$ surface (in this case the odd Chern classes vanish) and $n \leq 4$. For a partition $(n_1, \ldots, n_r)$ of $2n$ we write $(n_1, \ldots, n_r)$ for $c_{n_1}(S^{[n]}) \cdot \ldots \cdot c_{n_r}(S^{[n]})$. We obtain $(4) = 324$, $(2^2) = 828$, $(6) = 3200$, $(4,2) = 14720$ $(2^3) = 36800$, $(8) = 25650$, $(6,2) = 182340$, $(4^2) = 332730$, $(4,2^2) = 813240$, $(2^4) = 1992240$.

The strategy is similar to that of Remark 5.4: by Theorem 0.1 it is enough to compute the Chern numbers in case $S = \mathbb{P}_2$ and $S = \mathbb{P}_1 \times \mathbb{P}_1$. In both cases we have an action of a 2-dimensional torus Γ on $S^{[n]}$ with finitely many fix points. We proceed as above, again using a suitable Maple program.

It is remarkable that for any surface S and all $n \leq 7$ all the Chern numbers of $S^{[n]}$ are polynomials in $c_1(S)^2$ and $c_2(S)$ with nonnegative coefficients (this was observed by G. Thompson). G. Höhn has used these numbers to check the conjecture of [6] about the elliptic genus of the Hilbert scheme of points $S^{[n]}$ on a $K3$ surface for $n \leq 6$.

Finally, we compute the holomorphic Euler characteristics $\chi(F^{[n]})$ of the tautological bundles $F^{[n]}$ on $S^{[n]}$. Related results are shown, using different methods in the recent preprint [5].

Proposition 5.6 — *i) Let F be a vector bundle on S. Then*

$$\sum_{n,i} h^i(S^{[n]}, F^{[n]}) z^i t^{n-1} = \Big(\sum_j h^j(S,F) z^j\Big) \frac{(1+zt)^{h^1(\mathcal{O}_S)}}{(1-t)^{h^0(\mathcal{O}_S)}(1-z^2t)^{h^2(\mathcal{O}_S)}}.$$

In particular, if S is connected, then $h^0(S^{[n]}, F^{[n]}) = h^0(S, F)$, and if furthermore $h^i(S, \mathcal{O}_S) = 0$ for $i > 0$, then $h^i(S^{[n]}, F^{[n]}) = h^i(S, F)$ for all i.

ii) For all $F \in K(S)$ one has

$$\chi(F^{[n]}) = \chi(F)\binom{\chi(\mathcal{O}_S) + n - 2}{n-1}.$$

Proof. Part ii) follows from i) by putting $z = -1$. In order to prove part i) consider the cartesian diagram for $n \geq 1$

$$\begin{array}{ccc} \Sigma_n & \xrightarrow{\overline{f}} & Z_n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \overline{p}} \\ S^{[n]} & \xrightarrow{f} & S^{(n)}, \end{array}$$

where f is the Hilbert-Chow morphism and

$$Z_n := \left\{(\eta, x) \in S^{(n)} \times S \;\middle|\; x \in supp(\eta)\right\}$$

is the image of the universal family $\Sigma_n \subset S^{[n]} \times S$ in $S^{(n)} \times S$. We denote by $q : \Sigma_n \to S$, $\overline{q} : Z_n \to S$ the restriction of the projection. It is easy to see that there is an isomorphism $Z_n \cong S^{(n-1)} \times S$ which identifies $\overline{q}$ with the projection to the second factor. The maps p and $\overline{p}$ are finite. By proposition 1.2 the higher direct images $R^i f_* \mathcal{O}_{S^{[n]}}$ vanish. An easy spectral sequence argument shows that then the higher direct image sheaves $R^i \overline{f}_* \mathcal{O}_{\Sigma_n}$ must vanish as well. Moreover, $\overline{f}$ is a birational morphism of integral varities and Z_n is normal. This shows that $\overline{f}_* \mathcal{O}_{\Sigma_n} = \mathcal{O}_{Z_n}$. In particular, for any locally free sheaf G on Z_n one has $H^i(\Sigma_n, \overline{f}^* G) = H^i(Z_n, G)$.

By definition, $F^{[n]} = p_* q^* F$. As p is finite, we get

$$\begin{aligned} H^i(S^{[n]}, F^{[n]}) &\cong H^i(\Sigma_n, q^* F) = H^i(\Sigma_n, \overline{f}^* \overline{q}^* F) \\ &= H^i(Z_n, \overline{q}^* F) = H^i(S^{(n-1)} \times S, \mathcal{O}_{S^{(n-1)}} \boxtimes F) \end{aligned}$$

for all i. By the Künneth formula this gives

$$\sum_i h^i(S^{[n]}, F^{[n]}) z^i = \Big(\sum_i h^i(S, F) z^i\Big) \cdot \Big(\sum_i h^i(S^{(n-1)}, \mathcal{O}_{S^{(n-1)}}) z^i\Big).$$

By proposition 1.2, $h^i(S^{(n-1)}, \mathcal{O}_{S^{[n-1]}}) = h^i(S^{[n-1]}, \mathcal{O}_{S^{[n-1]}})$. Hence

$$\sum_n \sum_i h^i(S^{[n]}, F^{[n]}) z^i t^{n-1} = \Big(\sum_i h^i(S, F) z^i\Big) \cdot \Big(\sum_\nu \sum_i h^i(S^{[\nu]}, \mathcal{O}_{S^{[\nu]}}) z^i t^\nu\Big)$$

The second factor on the right hand side was computed in [12, Proposition 3.3] and equals

$$\frac{(1+zt)^{h^1(S, \mathcal{O}_S)}}{(1-t)^{h^0(S, \mathcal{O}_S)}(1-z^2 t)^{h^2(S, \mathcal{O}_S)}}.$$

□

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